Note on the 4- and 5-leaf powers*

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Abstract

Motivated by the problem of reconstructing evolutionary history, Nishimura et al. defined k-leaf powers as the class of graphs $G = (V, E)$ which has a k-leaf root T, i.e., T is a tree such that the vertices of G are exactly the leaves of T and two vertices in V are adjacent in G if and only if their distance in T is at most k. It is known that leaf powers are chordal graphs. Brandstädt and Le proved that every k-leaf power is a $(k+2)$-leaf power and every 3-leaf power is a k-leaf power for $k \geq 3$. They asked whether a k-leaf power is also a $(k+1)$-leaf power for any $k \geq 4$. Fellows et al. gave an example of a 4-leaf power which is not a 5-leaf power. It is interesting to find all the graphs which have both 4-leaf roots and 5-leaf roots. In this paper, we prove that, if G is a 4-leaf power with $L(G) \neq \emptyset$, then G is also a 5-leaf power, where $L(G)$ denotes the set of leaves of G.

Keywords: k-leaf power; k-leaf root; similar vertex; chordal graph; simplicial vertex

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1 Introduction

Motivated by the problem of reconstructing evolutionary history, Nishimura et al. [9] introduced the notion of k-leaf root and k-leaf power. Let $G = (V(G), E(G))$ be a finite undirected graph. For $k \geq 2$, a tree T is a *k-leaf root* of G if $V(G)$ is the set of leaves of T and two vertices $x, y \in V(G)$ are adjacent in G if and only if their distance $d_T(x, y)$ in T is at most k, i.e., $xy \in E(G)$ if and only if $d_T(x, y) \leq k$. G is called the *k-leaf power* of T.

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Obviously, a graph G is a 2-leaf power if and only if it is the disjoint union of cliques, i.e., G is P_3-free. The 3-leaf powers are exactly the bull-, dart-, and gem-free chordal graphs [7]; equivalently, 3-leaf powers are exactly the result of substituting cliques into the nodes of a tree. For more results on 3-leaf powers see [2, 10]. A characterization of 4-leaf powers in terms of forbidden subgraphs is much more complicated [3, 10]. For 5-leaf powers, a polynomial time recognition was given in [5] but no structural characterization of 5-leaf powers has been known. Recently, the authors of [11] characterized distance-hereditary 5-leaf powers (without pair of similar vertices) in terms of forbidden induced subgraphs. The complexity of characterizing and recognizing leaf powers in general is a major open problem, and so the complexity of characterizing and recognizing leaf powers for $k \geq 6$ is a major open problem.

It is well known that if a graph G has k-leaf roots for some k, then G is chordal. However, there are chordal graphs which are not k-leaf power for any $k \geq 2$. In [1], the authors showed that every interval graph is a leaf power and unit interval graphs are exactly the leaf powers which have a caterpillar as leaf root.

Brandstädt and Le [2] proved that every k-leaf power is a $(k+2)$-leaf power and every 3-leaf power is is a k-leaf power for $k \geq 3$. They asked whether k-leaf power is also a $(k+1)$-leaf power for any $k \geq 4$. Fellows et al. gave an example of a 4-leaf power which is not a 5-leaf power. Then it is interesting to find all the graphs which have both 4-leaf roots and 5-leaf roots. In this paper, we prove that, if G is a 4-leaf power with $L(G) \neq \emptyset$, then G is also a 5-leaf power, where $L(G)$ denotes the set of leaves of G.

2 Notations and basic facts

We consider $G = (V(G), E(G))$ as a finite, simple and undirected graph. For $k \geq 1$, let P_k denote a path with k vertices and $k-1$ edges, and, for $k \geq 3$, let C_k denote a cycle with k vertices and k edges. A vertex v is *pendent* or a *leaf* if its degree is 1, i.e., $d(v) = 1$. Let $L(G)$ denotes the set of leaves of G. A path $v_0v_1 \ldots v_t$ ($t \geq 1$) is *pendent* if $d(v_0) = 1$, $d(v_t) \geq 3$ and all other vertices have degree 2. Especially, if $t = 1$, it is called a *pendent edge*. The neighborhood of a vertex $u \in V(G)$ in the graph G is denoted by $N_G(u)$ and the closed neighborhood is denoted by $N_G[u] = \{u\} \cup N_G(u)$. Two vertices $u, v \in V(G)$ with $u \neq v$ are called *similar* if $N_G[u] = N_G[v]$. It is obvious that if u and v are a pair of similar vertices, then $uv \in E(G)$. An undirected graph is *chordal (triangulated, rigid circuit)* if every cycle of length greater than three has a chord, which is an edge connecting two nonconsecutive vertices on the cycle. Namely,

a graph is chordal if it contains no induced C_k for $k \geq 4$. A vertex is *simplicial* in G if its closed neighborhood $N[v]$ is a clique. It is well known that every chordal graph G has a simplicial vertex, furthermore, if G is not complete, then it has two nonadjacent simplicial vertices [6]. For various characterizations of chordal graphs, we refer to [4].

Firstly, we list some useful facts on leaf powers in Proposition 1 without proofs.

Proposition 1 *([2]) (i) For every $k \geq 2$, k-leaf powers are chordal.*
(ii) Every induced subgraph of a k-leaf power is a k-leaf power for $k \geq 2$.
(iii) A graph is a k-leaf power if and only if each of its connected components is a k-leaf power.

We mention some basic facts from [2, 10], and repeat their proofs for completeness.

Proposition 2 *([2]) (i) Every k-leaf power is a $(k+2)$-leaf power.*
(ii) Every 3-leaf power is a k-leaf power for $k \geq 3$.

Proof. (i) Let T be a k-leaf root of G, and let T' be the tree obtained from T by subdividing each pendant edge with a new vertex. Thus, the leaves of T' are exactly those of T. Clearly, for all $x, y \in V(G)$, $xy \in E(G)$ if and only if $k \geq d_T(x,y) = d_{T'}(x,y) - 2$, hence T' is a $(k+2)$-leaf of G.

(ii) Let T be a 3-leaf root of a graph G, and let T' be the tree obtained from T by subdividing each non-pendent edge with exactly $k-3$ new vertices. Thus, the leaves of T' are exactly those of T. Clearly, for all $x, y \in V(G)$, $xy \in E(G)$ if and only if $d_T(x,y) = d_{T'}(x,y) = 2$, or $d_T(x,y) = 3$ and $d_{T'}(x,y) = k$. Hence T' is a k-leaf root of G. ∎

Proposition 3 *([10]) If $G = (V, E)$ is a graph and $u, v \in V$ are similar, then G has a k-leaf root if and only if $G - u$ has a k-leaf root for any $k \geq 2$.*

Proof. If G has a k-leaf root T, denoting T' the tree obtained from T by deleting the leaf u, i.e., the pendent path containing u in T, then T' is a k-leaf root of $G - u$.

If $G - u$ has a k-leaf root T', we can obtained a leat root T of G by attaching the new leaf u to the neighbor of the leaf v in T'. ∎

Theorem 1 ([10]) *Let $G = (V, E)$ be a graph without pairs of similar vertices. Then G has a 4-leaf root if and only if it is chordal and does not contain any of the graphs in Figure 1 as an induced subgraph.*

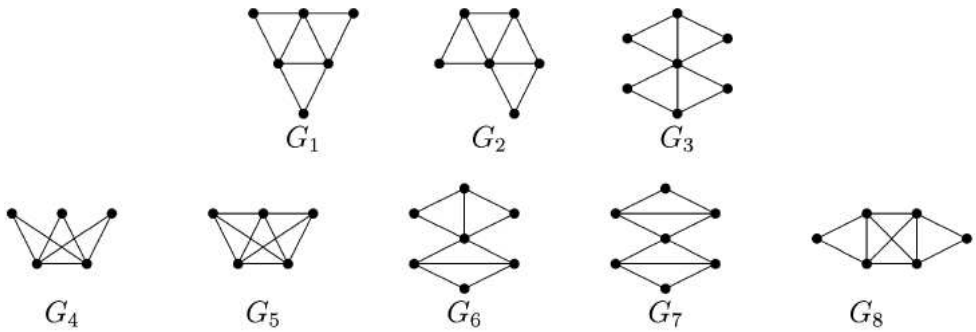

Figure 1 Forbidden subgraphs of 4-leaf powers without similar vertices.

3 Main results

We define a class of simple graphs $\mathcal{H}$ satisfying that: $H \in \mathcal{H}$, if H has a clique C with vertex set $\{c_1, c_2, \ldots, c_t\}$ such that there is a leaf u adjacent to one vertex c_1 of C and $|N_H(v) \cap C| \leq 1$ for all $v \in H \setminus \{u, c_1, \ldots, c_t\}$, $t \geq 2$. Let T_0 be a tree (as shown in Figure 2) satisfying that $d_{T_0}(c_1, u) = 5$, $d_{T_0}(c_1, c_i) = 5$ for $2 \leq i \leq t$ and $d_{T_0}(c_i, c_j) = 4$ for all $2 \leq i, j \leq t$ and $i \neq j$ if $t \geq 3$.

Lemma 1 *For any graph $H \in \mathcal{H}$, suppose H is a 5-leaf power without pairs of similar vertices, then there must be a 5-leaf root T of H containing T_0 as its subtree.*

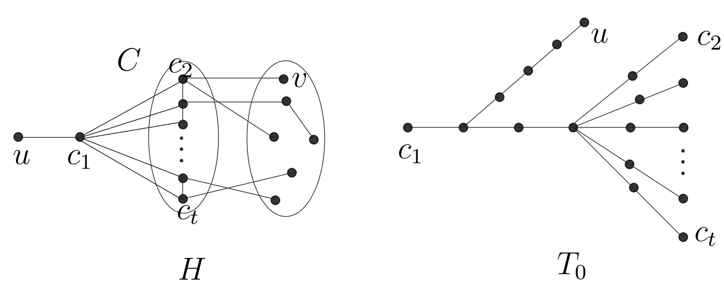

Figure 2 One 5-leaf root of a graph $H \in \mathcal{H}$.

Proof. Let $H \in \mathcal{H}$ be a 5-leaf power without pairs of similar vertices. We can assume that in each 5-leaf root T of H, the pendent path ending at the leaf u must have length 4 and connect to the neighbor of c_1 in T. Otherwise, since u is a leaf of H, we can attain what we need by subdividing the leaf edge and moving the pendent paths ending at the leaf u to the neighbor of c_1. Thus, suppose T is a 5-leaf root of H satisfying

the above conditions. If T contains T_0 as a subtree, we are done. Otherwise, we will construct a new tree T' from T, which is a 5-leaf root of H and contains T_0 as a subtree.

If $t = 2$, we have $d_T(c_1, c_2) \geq 3$. Since T does not contain T_0 as a subtree, then $d_T(c_1, c_2) = 3$ or 4. Since H have no pairs of similar vertices, then in these two cases, the pendent path containing u in T must connect to the neighbor of c_1. Let c_1abc_2 (or c_1abcc_2) be the path joining vertices c_1 and c_2 in T. Denote by T' the new tree constructed from T by subdividing edge ab with exactly two (or one) new vertices. It is easy to verify that T' is a 5-leaf root of H, which contains T_0 as a subtree. Thus, in the following we always assume $t \geq 3$.

If for each $2 \leq i \leq t$, $d_T(c_1, c_i) = 3$, without loss of generality, let c_1abc_2 be the path joining vertices c_1 and c_2 in T. Since H has no pairs of similar vertices, we notice that in T there are no pendent paths containing c_i ($i \geq 3$) and connecting to the vertex b and the length of each pendent path containing c_i ($i \geq 3$) and connecting to the vertex a must be 2. Denote by T' the new tree constructed from T by subdividing edge c_1a with exactly two new vertices, deleting the pendent path containing u and attaching a pendent path with leaf u to the neighbor of c_1 such that $d_{T'}(c_1, u) = 5$. It is easy to verify that $d_{T'}(c_1, u) = 5$, $d_{T'}(c_1, c_i) = 5$ for $2 \leq i \leq t$ and $d_{T'}(c_i, c_j) = 4$ for all $2 \leq i, j \leq t$ and $i \neq j$. Therefore, T' is a 5-leaf root of H, which contains T_0 as a subtree.

Assume $\max_{2 \leq i \leq t} d_T(c_1, c_i) = 4$, without loss of generality, suppose $d_T(c_1, c_2) = 4$. Let c_1abcc_2 be the path joining vertices c_1 and c_2 in T, then there are no pendent paths containing c_i ($i \geq 3$) and connecting to the vertex c. Let P_{ai} (or P_{bi}) be the pendent paths containing c_i and connecting to the vertex a (or b). Since $\max_{2 \leq i \leq t} d_T(c_1, c_i) = 4$ and C is a clique, the length of P_{ai} (or P_{bi}) is at most 2. Since H has no pairs of similar vertices, the length of each P_{ai} is exactly two and there is at most one pendent path with length one among all pendent paths P_{bi} ($i \geq 3$). Without loss of generality, let $|P_{b3}| = 1$. Denote by T' the new tree obtained from T by subdividing edge bc_3 with exactly one new vertex, then moving all the pendent paths P_{ai} ($i \geq 4$, if they exist) to the vertex b, and then subdividing edge ab with exactly one new vertex. We can verify that T' is a 5-leaf root of H, which contains T_0 as a subtree. Firstly, we notice that for all $2 \leq i, j \leq t$ and $i \neq j$, $d_{T'}(c_1, c_i) = 5$, $d_{T'}(c_i, c_j) = 4$. For each vertex $v \in H \setminus \{u, c_1, \ldots, c_t\}$ such that $d_{T'}(v, c_i) \leq 5$, there exists no vertex c_j ($j \neq i$) such that $d_{T'}(v, c_j) \leq 5$. Otherwise, we have $d_T(v, c_1) \leq 5$ or $d_T(v, c_j) \leq 5$, contradicting to the definition of $\mathcal{H}$.

For the case $\max_{2 \leq i \leq t} d_T(c_1, c_i) = 5$, we also can construct a new tree T' from T by the similar transformations as above, which contains T_0 as a subtree. ∎

Theorem 2 *If G is a 4-leaf power with $L(G) \neq \emptyset$, then G is also a 5-leaf power, where $L(G)$ denotes the set of leaves of G.*

Proof. In view of Proposition 1 (iii) and Proposition 3, it suffices to consider connected graphs without pairs of similar vertices.

By contrary, we assume $G = (V, E)$ is a 4-leaf power without pairs of similar vertices such that $L(G) \neq \emptyset$, and G is not a 5-leaf power. Furthermore, we assume that among all such graphs G is chosen such that it has minimum number of edges and subject to this condition it has maximum number of leaves.

Let u be a leaf of G and v_1 the unique neighbor of u. If $G' = G - u$ has no pair of similar vertices, then G' has a 5-leaf root T'. Thus, we can obtain a 5-leaf root of G by attaching a pendent path of length 4 ending at the new leaf u to the neighbor of the leaf v_1 in T'.

Hence, we may assume that G' has a pair of similar vertices. Note that v_1 must be one of each pair of similar vertices, since v_1 is the only vertex whose neighborhood is changed in G'. In fact, there is a unique vertex $v_2 \in V$ such that v_1 and v_2 are similar in G'. For otherwise, suppose v_1 and v_3 ($v_3 \neq v_2$) are also similar, then by the construction of G', we can obtain that v_2 and v_3 are similar in G. Since G does not contain G_4 and G_5 (see Figure 1) as an induced subgraph, the set $N = N_G(v_1) \setminus \{u, v_2\}$ induces either a complete graph or the disjoint union of two complete graphs.

Case 1. N induces a complete graph.

It is easy to observe that $G - v_2$ is connected and has no similar vertices. Then $G - v_2$ has a 5-leaf root T'. If there exists a vertex $w \in N$ such that $d_{T'}(v_1, w) = 5$, let v_1abcdw be the path in T' joining vertices v_1 and w. We can construct a 5-leaf root T of G from T' by attaching a path with length 2 ending at a new leaf v_2 to the vertex c.

In the following, we assume $d_{T'}(w, v_1) \leq 4$ for all $w \in N$. If there exists a vertex $w \in N$ such that $d_{T'}(v_1, w) = 4$, let v_1abcw be the path in T' joining vertices v_1 and w. We can construct a 5-leaf root T of G from T' by attaching a pendent path with length 2 ending at a new leaf v_2 to the vertex b. Similarly, if $d_{T'}(w, v_1) = 3$ for all vertices $w \in N$, then attaching a path of length 2 ending at the new leaf v_2 to the neighbor of the leaf v_1 in T' yields a 5-leaf root of G. These contradictions complete the proof of **Case 1**.

Case 2. N induces the disjoint union of two complete graphs N_1 and N_2.

Since G is a 4-leaf power, G is chordal by Proposition 1. Thus the graph $G - \{u, v_1, v_2\}$ has exactly two components with vertex sets U_1 and U_2 satisfying that

$N_1 \subseteq U_1$ and $N_2 \subseteq U_2$. Note that for $i = 1, 2$, $|N_i| \geq 1$.

Subcase 2.1. $|U_1| \geq 2$, $|U_2| \geq 2$.

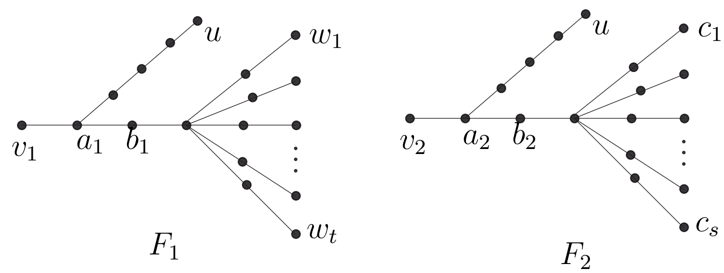

Figure 3 5-leaf roots of graphs H_1 and H_2 in Subcase 2.1.

Since $|U_i| \geq 2$ for $i = 1, 2$ and G has no pair of similar vertices, we have $U_i \setminus N_i \neq \emptyset$. For any vertex $x_i \in U_i \setminus N_i$ $(i = 1, 2)$, $|N_G(x_i) \cap N_i| \leq 1$. If $|N_i| = 1$ for $i = 1, 2$, we are done. Suppose $|N_i| \geq 2$, let $N_1 = \{w_1, w_2, \ldots, w_t\}$ and $N_2 = \{c_1, c_2, \ldots, c_s\}$. Suppose for some i, there is a vertex $x_i \in U_i \setminus N_i$ and $|N_G(x_i) \cap N_i| \geq 2$. Then G contains G_8 (see Figure 1) as a subgraph, a contradiction. Let $H_1 = G - U_2 - v_2$ and $H_2 = G - U_1 - v_1 + uv_2$. For $i = 1, 2$, H_i is a 5-leaf power without pairs of similar vertices by the choice of the graph G. It is obvious that $H_1, H_2 \in \mathcal{H}$. By Lemma 1, there must be a 5-leaf root T_i of H_i containing T_0 as its subtree, denote by F_i the subtree of T_i which is isomorphic to T_0, as shown in Figure 3. We can construct a new tree T from T_1 and T_2 by deleting the pendent path containing u in T_2, adding an edge b_1b_2 and contracting edges a_1b_1 and a_2b_2. It is easy to verify that T is a 5-leaf root of G, a contradiction.

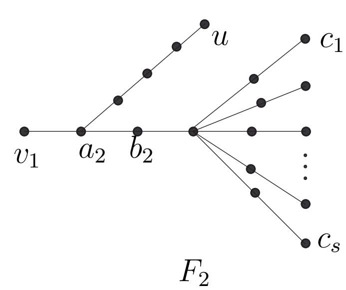

Figure 4 5-leaf root of graph H_2 in Subcase 2.2.

Subcase 2.2. $|U_1| = 1$, $|U_2| \geq 2$.

Suppose $U_1 = \{w_1\}$, $N_2 = \{c_1, c_2, \ldots, c_s\}$. By similar discussions as Subcase 2.1, we know that $U_2 \setminus N_2 \neq \emptyset$ and for any vertex $x \in U_2 \setminus N_2$, $|N_G(x) \cap N_2| \leq 1$. Let $H_2 = G - w_1 - v_2$. Since H_2 has no pairs of similar vertices, $H_2 \in \mathcal{H}$. By Lemma 1, H_2 has a 5-leaf root T_2 containing T_0 as its subtree, denote by F_2 the subtree of T_2 which is isomorphic to T_0, as shown in Figure 4. Now we also can construct a 5-leaf root T of G by attaching a pendent path with length 2 ending at a new leaf v_2 to the vertex b_2 and attaching a pendent path with length 2 ending at a leaf w_1 to the vertex a_2, a contradiction. ∎

From Proposition 2 (i) and Theorem 2, we conclude the following corollary.

Corollary 1 *If G is a 4-leaf power with $L(G) \neq \emptyset$, then G is also a k-leaf power for $k \geq 4$.*

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